

# BOUNDING SCALAR CURVATURE FOR GLOBAL SOLUTIONS OF THE KÄHLER-RICCI FLOW

JIAN SONG AND GANG TIAN

**ABSTRACT.** We show that the scalar curvature is uniformly bounded for the normalized Kähler-Ricci flow on a Kähler manifold with semi-ample canonical bundle. In particular, the normalized Kähler-Ricci flow has long time existence if and only if the scalar curvature is uniformly bounded, for Kähler surfaces, projective manifolds of complex dimension three, and for projective manifolds of all dimensions if assuming the abundance conjecture.

## 1. INTRODUCTION

Let  $(X, g_0)$  be a Kähler manifold of  $\dim_{\mathbb{C}} X = n \geq 2$  and  $g_0$  a smooth Kähler metric. We consider the unnormalized Kähler-Ricci flow

$$(1.1) \quad \frac{\partial g}{\partial t} = -Ric(g), \quad g|_{t=0} = g_0.$$

By definition,  $X$  is a minimal model if its canonical line bundle  $K_X$  is nef. It is well-known [Ts2, TiZha] that the flow has a global solution on  $X \times [0, \infty)$  if and only if the canonical line bundle  $K_X$  is nef or equivalently,  $X$  is a minimal model. The abundance conjecture predicts that if the canonical line bundle  $K_X$  over a projective manifold  $X$  is nef, then it must be semi-ample, i.e., a sufficiently large power of  $K_X$  is globally generated or base point free. In particular, the abundance conjecture holds for projective manifolds of complex dimension no bigger than three [Ka, M1, M2]. The aim of this paper is to investigate the behavior of the scalar curvature along the Kähler-Ricci flow on a Kähler manifold with semi-ample canonical line bundle.

**Theorem 1.1.** *Let  $X$  be a Kähler manifold with  $K_X$  being semi-ample, and  $g(t)$  be the smooth global solution of the normalized Kähler-Ricci flow*

$$(1.2) \quad \frac{\partial g}{\partial t} = -Ric(g) - g, \quad g|_{t=0} = g_0.$$

*Then there exists  $C > 0$  depending on  $X$  and  $g_0$ , such that for all  $t \in [0, \infty)$ ,*

$$(1.3) \quad |R(t)|_{L^\infty(X)} \leq C,$$

*where  $R(t)$  is the scalar curvature of  $g(t)$ .*

If we assume the abundance conjecture, then Theorem 1.1 holds for all projective minimal models, and it implies that the long time existence of the normalized Kähler-Ricci flow over a projective manifold is equivalent to the scalar curvature being uniformly bounded. In the case when  $X$  is a minimal model of general type,

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i.e.,  $K_X$  is big and semi-ample, the uniform scalar curvature bound is proved by Zhang [Z2] and so the normalized Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric with uniformly bounded scalar curvature.

When  $0 < \text{kod}(X) < n$ , a minimal model  $X$  with semi-ample  $K_X$  admits a Calabi-Yau fibration over its canonical model  $X_{\text{can}}$ , it is shown in [ST1, ST2] that the normalized Kähler-Ricci flow collapses nonsingular Calabi-Yau fibres and the flow converges weakly to a generalized Kähler-Einstein metric  $g_{\text{can}}$  on its canonical model  $X_{\text{can}}$ . In particular, on a dense Zariski open set of  $X_{\text{can}}$ , such a canonical metric satisfies the generalized Einstein equation

$$(1.4) \quad \text{Ric}(g_{\text{can}}) = -g_{\text{can}} + g_{WP},$$

where  $g_{WP}$  is the Weil-Petersson metric induced from the Calabi-Yau fibration from  $X$  to  $X_{\text{can}}$ . Then Theorem 1.1 shows that in this case, the normalized Kähler-Ricci flow collapses the Calabi-Yau fibration with uniformly bounded scalar curvature. In particular, it improves the result in [ST1] for bounding the scalar curvature for the Kähler-Ricci flow on minimal elliptic surfaces.

When  $\text{kod}(X) = 0$ ,  $X$  is a Calabi-Yau manifold and it is proved in [C] that the unnormalized Kähler-Ricci flow converges to the unique Ricci-flat Kähler metric exponentially fast.

After rescaling time and space simultaneously, we have the following immediate corollary from Theorem 1.1 for bounding the scalar curvature along the unnormalized Kähler-ricci flow.

**Corollary 1.1.** *Let  $X$  be a Kähler manifold with  $K_X$  being semi-ample, and  $g(t)$  be the smooth global solution of the unnormalized Kähler-Ricci flow (1.1). Then there exists  $C > 0$  depending on  $X$  and the initial Kähler metric, such that for all  $t \in [0, \infty)$ ,*

$$(1.5) \quad |R(t)|_{L^\infty(X)} \leq C(1+t)^{-1}.$$

For  $X$  of  $\text{kod}(X) > 0$ , (1.5) is optimal in the sense that there exists  $c > 0$  such that for all  $t \geq 0$ , there exists  $z_t \in X$  such that  $|R(t, z_t)| \geq c(1+t)^{-1}$ . This can be seen from the simple example  $X = E \times C$  where  $E$  is an elliptic curve and  $C$  is a curve with genus greater than 1.

We also give a criterion for long time existence of the normalized Kähler-Ricci flow for Kähler surfaces and projective manifolds of dimension 3.

**Corollary 1.2.** *Let  $X$  be a Kähler surface or a projective manifold of complex dimension 3. Then the normalized Kähler-Ricci flow (1.2) on  $X$  admits a global solution if and only if the scalar curvature is uniformly bounded in time.*

In general, it is natural to ask if the following holds for the maximal solution of the unnormalized Kähler-Ricci flow on  $X \times [0, T)$ , where  $X$  is a Kähler manifold and  $T > 0$  is the maximal existence time.

- (1) If  $T < \infty$ , then there exists  $C > 0$  such that

$$-C \leq R(t) \leq C(T-t)^{-1}.$$

- (2) If  $T = \infty$ , then there exists  $C > 0$  such that

$$|R(t)| \leq C(1+t)^{-1}.$$

In [Pe2, SeT], the answer to the first question is affirmative due to Perelman for the Kähler-Ricci flow on Fano manifolds with finite time extinction. In [Z3], it is shown that if the Kähler-Ricci flow develops finite time singularity, the scalar curvature blows up at most of rate  $(T-t)^{-2}$  if  $X$  is projective and if the initial Kähler class lies in  $H^2(X, \mathbb{Q})$ . One can even ask if the above estimates hold for the Ricci curvature along the unnormalized Kähler-Ricci flow.

## 2. VOLUME ESTIMATES AND PARABOLIC SCHWARZ LEMMA

Let  $X$  be an  $n$ -dimensional Kähler manifold with  $K_X$  being semi-ample. Therefore the canonical ring  $R(X, K_X)$  is finitely generated, and so the pluricanonical system  $|mK_X|$  for sufficiently large  $m \in \mathbb{Z}^+$ , induces a holomorphic map

$$(2.6) \quad \pi : X \rightarrow X_{can} \subset \mathbb{P}^N,$$

where  $X_{can}$  is the canonical model of  $X$ . The Kodaira dimension of  $X$  is defined to be

$$(2.7) \quad \text{kod}(X) = \dim X_{can}.$$

We always have

$$0 \leq \text{kod}(X) \leq \dim X = n.$$

In particular,

- (1) If  $\text{kod}(X) = n$ ,  $X$  is birationally equivalent to its canonical model  $X_{can}$ , and  $X$  is called a minimal model of general type.
- (2) If  $0 < \text{kod}(X) < n$ ,  $X$  admits a Calabi-Yau fibration

$$\pi : X \rightarrow X_{can}$$

over  $X_{can}$  and a general fibre is a smooth Calabi-Yau manifold of complex dimension  $n - \text{kod}(X)$ .

- (3) If  $\text{kod}(X) = 0$ ,  $X_{can}$  is a point and  $X$  is a Calabi-Yau manifold with  $c_1(X) = 0$ .

Now we will reduce the normalized Kähler-Ricci flow to a parabolic Monge-Ampère equation. Let  $\mathcal{O}_{\mathbb{P}^N}(1)$  be the hyperplane bundle over  $\mathbb{P}^N$  in (2.6) and  $\omega_{FS} \in [\mathcal{O}_{\mathbb{P}^N}(1)]$  be a Fubini-Study metric on  $\mathbb{P}^N$ . Then there exists  $m > 0$  such that

$$mK_X = \pi^* \mathcal{O}_{\mathbb{P}^N}(1).$$

We define

$$\chi = \frac{1}{m} \pi^* \omega_{FS} \in [K_X]$$

and  $\chi$  is a smooth nonnegative closed  $(1, 1)$ -form on  $X$ . There also exists a smooth volume form  $\Omega$  on  $X$  such that

$$\text{Ric}(\Omega) = -\sqrt{-1} \partial \bar{\partial} \log \Omega = \chi.$$

Let  $\omega_0$  be the initial Kähler metric of the normalized Kähler-Ricci flow (1.2) on  $X$ . Then Kähler class evolving along the normalized Kähler-Ricci flow is given by

$$[\omega(t)] = (1 - e^{-t})[K_X] + e^{-t}[\omega_0]$$

and so  $[\omega(t)]$  is a Kähler class for all  $t \in [0, \infty)$ . Therefore the normalized Kähler-Ricci flow starting with  $\omega_0$  on  $X$  has a smooth global solution on  $X \times [0, \infty)$ . We define the reference metric

$$\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0.$$

Then the Kahler-Ricci flow is equivalent to the following Monge-Ampere flow.

$$(2.8) \quad \frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\kappa)t}(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi,$$

where  $\omega_t = \chi + e^{-t}(\omega_0 - \chi)$  and  $\kappa = \text{cod}(X)$ . In particular,  $\chi = 0$  when  $\text{cod}(X) = 0$ .

First we prove the following uniform estimates.

**Lemma 2.1.** *There exists  $C > 0$  such that on  $X \times [0, \infty)$ ,*

$$(2.9) \quad \frac{\partial \varphi}{\partial t} \leq C$$

and

$$(2.10) \quad |\varphi| \leq C.$$

*Proof.* It is straightforward to show that there exists  $C_1 > 0$  such that

$$C_1^{-1}e^{-nt}\Omega \leq \omega_t^n \leq C_1e^{-(n-\kappa)t}\Omega.$$

Then by applying the maximum principle for  $\varphi$  and  $e^{-t}\varphi$ , there exists  $C_2 > 0$  such that on  $X \times [0, \infty)$ ,

$$\varphi \leq C_2, \quad |e^{-t}\varphi| \leq C_2.$$

Straightforward calculations show that

$$(2.11) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \frac{\partial \varphi}{\partial t} = -e^{-t} \text{tr}_\omega(\omega_0 - \chi) - \frac{\partial \varphi}{\partial t} + (n - \kappa).$$

The uniform upper bound for  $\frac{\partial \varphi}{\partial t}$  follows by applying the maximum principle to  $\frac{\partial \varphi}{\partial t} - e^{-t}\varphi$  since there exists  $C_3 > 0$  such that

$$\begin{aligned} & \left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{\partial \varphi}{\partial t} - e^{-t}\varphi \right) \\ &= -e^{-t} \text{tr}_\omega(\omega_t + \omega_0 - \chi) - (1 + e^{-t}) \frac{\partial \varphi}{\partial t} + e^{-t}\varphi + ne^{-t} + (n - \kappa) \\ &\leq -\frac{\partial \varphi}{\partial t} + C_3. \end{aligned}$$

Then we immediately obtain the uniform upper bound for  $\frac{\partial \varphi}{\partial t}$ .

Now we will prove the lower bound for  $\varphi$ . Rewrite the the parabolic Monge-Ampère equation as

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-(n-\kappa)t+\varphi+\frac{\partial \varphi}{\partial t}}\Omega.$$

Since  $\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0$  and there exists  $C_4 > 0$  such that  $C_4^{-1} \leq e^{(n-\kappa)t}[\omega_t]^n \leq C_4$ ,  $\varphi + \frac{\partial \varphi}{\partial t} \leq C_4$ , by the results of [DP, EGZ2], there exists  $C_5 > 0$  such that for all  $t \in [0, \infty)$ ,

$$(2.12) \quad \sup_{z \in X} \varphi(z, t) - \inf_{z \in X} \varphi(z, t) \leq C_5.$$

On the other hand,

$$\int_X e^{\varphi + \frac{\partial \varphi}{\partial t}} \Omega = e^{(n-\kappa)t} [\omega_t]^n \geq C_4^{-1},$$

and so there exists  $C_6 > 0$  such that for all  $t \in [0, \infty)$ ,

$$\sup_{z \in X} \left( \varphi + \frac{\partial \varphi}{\partial t} \right) \geq -C_6.$$

Hence

$$\sup_{z \in X} \varphi(z, t) \geq \sup_{z \in X} \left( \varphi(z, t) + \frac{\partial \varphi}{\partial t}(z, t) \right) - \sup_{z \in X} \frac{\partial \varphi}{\partial t}(z, t) \geq - \sup_{z \in X} \frac{\partial \varphi}{\partial t}(z, t) - C_6$$

and so  $\sup_{z \in X} \varphi(z, t)$  is uniformly bounded from below for all  $t \in [0, \infty)$ . Then (2.10) is proved by applying the estimate (2.12).  $\square$

We now shall prove a uniform bound for  $\frac{\partial \varphi}{\partial t}$ .

**Proposition 2.1.** *There exists  $C > 0$  such that ,*

$$(2.13) \quad \left| \frac{\partial \varphi}{\partial t} \right|_{L^\infty(X \times [0, \infty))} \leq C.$$

*Proof.* It suffices to prove a uniform lower bound for  $\frac{\partial \varphi}{\partial t}$  by Lemma 2.1.

First we consider the following family of Monge-Ampère equations for  $s \in [0, \infty)$ ,

$$(2.14) \quad (\omega_s + \sqrt{-1} \partial \bar{\partial} \psi_s)^n = e^{\psi_s} e^{-(n-\kappa)s} \Omega,$$

where

$$\omega_s = \chi + e^{-s}(\omega_0 - \chi).$$

There exists a unique smooth solution  $\psi_s$  for each  $s \in [0, \infty)$  [Y1, A]. It is straightforward to show by the maximum principle, that  $\psi_s$  is uniformly bounded above on  $X$  for all  $s \in [0, \infty)$ , i.e., there exists  $C_1 > 0$  such that

$$\psi_s \leq C_1$$

for all  $s \in [0, \infty)$ . Also

$$\int_X e^{\psi_s} \Omega = e^{(n-\kappa)s} [\hat{\omega}_s]^n$$

is uniformly bounded from below for all  $s \in [0, \infty)$ . Therefore there exists  $C_2 > 0$  such that

$$-C_2 \leq \sup_X \psi_s \leq C_2$$

for all  $s \in [0, \infty)$ .

By the results due to [DP, EGZ2] for solutions of degenerate Monge-Ampère equations, there exists  $C_3 > 0$  such that

$$|\psi_s|_{L^\infty(X)} \leq C_3$$

for all  $s \in [0, \infty)$ .

Let  $\rho = \rho(t)$  be a smooth decreasing function defined on  $[0, 1]$  such that

$$(2.15) \quad \rho(t) = \begin{cases} 1, & t \in [0, \frac{1}{3}] \\ 0, & t \in [\frac{2}{3}, 1]. \end{cases}$$

We now define a smooth function  $\Phi(z, t)$  on  $X \times [0, \infty)$  as follows.

$$(2.16) \quad \Phi(z, t) = \rho(t - m)\psi_{m+1}(z) + (1 - \rho(t - m))\psi_{m+2}(z), \quad (z, t) \in X \times [m, m + 1],$$

where  $m$  is any nonnegative integer. Then  $\Phi$  is smooth in  $X \times [0, \infty)$  and

$$-C_3 \leq \Phi \leq C_3.$$

Consider the same family of Monge-Ampère equations as in equation (2.14) for all  $s \in [1, \infty)$  and define  $\Phi$  as in equation (2.16). Now we consider

$$H = \frac{\partial \varphi}{\partial t} + 2\varphi - \Phi.$$

The evolution of  $H$  is given by

$$\left( \frac{\partial}{\partial t} - \Delta \right) H = 2 \frac{\partial \varphi}{\partial t} + tr_{\omega} (\chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \Phi) - (n + \kappa) - \frac{\partial \Phi}{\partial t},$$

and there exists  $C_4 > 0$  such that,

$$\left( \frac{\partial}{\partial t} - \Delta \right) H \geq 2 \log \frac{e^{(n-\kappa)t} \omega^n}{\Omega} + tr_{\omega} (\chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \Phi) - C_4.$$

It is straightforward to check that for all  $t \in [m, m + 1]$ ,

$$\chi + \omega_t \geq \omega_{m+1}, \quad \chi + \omega_t \geq \omega_{m+2}.$$

Suppose  $t \in [m, m + 1]$  for some nonnegative integer  $m$ . Then either  $\rho(t - m) \geq 1/2$  or  $(1 - \rho(t - m)) \geq 1/2$  and we have

$$\begin{aligned} \chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \Phi &\geq \frac{1}{2} \min (\chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \psi_{m+1}, \chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \psi_{m+2}) \\ &\geq \frac{1}{2} \min (\omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \psi_{m+1}, \omega_{m+2} + \sqrt{-1} \partial \bar{\partial} \psi_{m+2}) \end{aligned}$$

and so there exist  $C_3, C_4 > 0$  such that

$$\begin{aligned} (\chi + \omega_t + \sqrt{-1} \partial \bar{\partial} \Phi)^n &\geq 2^{-n} \min ((\omega_{m+1} + \sqrt{-1} \partial \bar{\partial} \psi_{m+1})^n, (\omega_{m+2} + \sqrt{-1} \partial \bar{\partial} \psi_{m+2})^n) \\ &\geq C_5 e^{-(n-\kappa)(m+2)} \Omega \\ &= C_5 e^{-(n-\kappa)(m+2-t)} e^{-(n-\kappa)t} \Omega \\ &\geq C_6 e^{-(n-\kappa)t} \Omega. \end{aligned}$$

Suppose that

$$H(z_0, t_0) = \inf_{X \times [0, T]} H(z, t)$$

with  $t_0 \in [m, m+1] \cap [0, T]$  for some nonnegative integer  $m$ . By combining the above estimates, we have at  $(z_0, t_0)$ ,

$$\begin{aligned}
 \left( \frac{\partial}{\partial t} - \Delta \right) H &\geq 2 \log \frac{e^{(n-\kappa)t_0} \omega^n}{\Omega} + C_7 \left( \frac{(\omega_{t_0} + \sqrt{-1} \partial \bar{\partial} \Phi)^n}{\omega^n} \right)^{1/n} - C_8 \\
 &\geq 2 \log \frac{e^{(n-\kappa)t_0} \omega^n}{\Omega} + C_7 \left( \frac{e^{-(n-\kappa)t_0} \Omega}{\omega^n} \right)^{1/n} - C_8 \\
 &\geq C_9 \left( \frac{e^{-(n-\kappa)t_0} \Omega}{\omega^n} \right)^{1/n} - C_{10} \\
 &\geq C_{11} e^{-H/n} - C_{10},
 \end{aligned}$$

where the last inequality follows from the fact that  $\varphi$  and  $\Phi$  are uniformly bounded. By the maximum principle,  $H(z_0, t_0) \geq n \log(C_{10}^{-1} C_{11})$  and so for all  $(z, t)$ ,

$$H(z, t) \geq \min \left( \min_X H(z, 0), n \log(C_{10}^{-1} C_{11}) \right),$$

and so it is uniformly bounded below. The proposition follows then immediately because  $\varphi$  and  $\Phi$  are uniformly bounded.  $\square$

The following calculation for the parabolic Schwarz lemma is given in [ST1, ST2].

**Lemma 2.2.** *Let  $\omega = \omega(t)$  be the solution of the normalized Kähler-Ricci flow (1.2). If  $\text{kod}(X) > 0$ , then there exists  $C > 0$  such that on  $X \times [0, \infty)$ ,*

$$(2.17) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_\omega(\chi) \leq \text{tr}_\omega(\chi) + C (\text{tr}_\omega(\chi))^2 - |\nabla \text{tr}_\omega(\chi)|_g^2,$$

where  $\Delta$  is the Laplace operator associated to the evolving metric  $g(t)$ .

The following proposition is an improvement in [ST1, ST2]. The fact that  $\frac{\partial \varphi}{\partial t}$  is bounded below helps to get the Schwarz lemma and the lower bound of  $\omega$  by  $\chi$ .

**Proposition 2.2.** *There exists  $C > 0$  such that on  $X \times [0, \infty)$ ,*

$$(2.18) \quad \text{tr}_\omega(\chi) \leq C.$$

*Proof.* Let  $H = \log \text{tr}_\omega \chi - A\varphi$ . Applying Lemma 2.2, there exist  $C_1, C_2, C_3 > 0$  such that

$$\left( \frac{\partial}{\partial t} - \Delta \right) H \leq -\text{tr}_\omega(A\omega_t - C_1\chi) - C_2 \leq -\text{tr}_\omega(\chi) - C_2 \leq -C_3 e^{-H} - C_2.$$

By the maximum principle,  $H$  is uniformly bounded and the proposition follows immediately.  $\square$

## 3. GRADIENT ESTIMATES

In this section, we will make use of the volume bound and parabolic Schwarz lemma to prove a parabolic analogue of Yau's gradient estimate [LiYa, ChY], to bound the scalar curvature. The parabolic gradient estimate is applied by Perelman to bound the scalar curvature for the Kähler-Ricci flow on Fano manifolds [Pe2, SeT]. This is also an improvement of in [ST1, ST2].

We consider the normalized parabolic Monge-Ampère flow (2.8) and let  $u = \frac{\partial \varphi}{\partial t} + \varphi$ . Since both  $\frac{\partial \varphi}{\partial t}$  and  $\varphi$  are uniformly bounded, there exists  $A > 0$  such that

$$A - u \geq 1.$$

**Proposition 3.1.** *There exists  $C > 0$  such that*

$$(3.19) \quad |\nabla u|_g^2 \leq C,$$

$$(3.20) \quad -\Delta u \leq C.$$

*Proof.* The proof is adapted from the calculations in [ST1, ST2]. We assume that  $\text{kod}(X) > 0$ . When  $\text{kod}(X) = 0$ , the proof of the proposition follows the same way since  $\chi = 0$  and it is in fact simpler.

First we note that

$$\left(\frac{\partial}{\partial t} - \Delta\right)u = \text{tr}_\omega(\chi) - \kappa.$$

The evolution for  $|\nabla u|_g^2$  and  $\Delta u$  are given as below where  $|\nabla u|_g^2 = g^{i\bar{j}}u_i u_{\bar{j}}$  and  $g = g(t)$  is the evolving metric associated to  $\omega(t)$ .

$$(3.21) \quad \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|_g^2 = |\nabla u|_g^2 + (\nabla \text{tr}_\omega(\chi) \cdot \bar{\nabla} u + \bar{\nabla} \text{tr}_\omega(\chi) \cdot \nabla u) - |\nabla \nabla u|_g^2 - |\bar{\nabla} \nabla u|_g^2,$$

$$(3.22) \quad \left(\frac{\partial}{\partial t} - \Delta\right)\Delta u = \Delta u + g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}u_{i\bar{j}} + \Delta \text{tr}_\omega(\chi).$$

Let

$$H = \frac{|\nabla u|_g^2}{A - u} + \text{tr}_\omega(\chi).$$

Then we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)H \\ &= \left(\frac{|\nabla u|_g^2 - |\nabla \nabla u|_g^2 - |\bar{\nabla} \nabla u|_g^2 + (\nabla \text{tr}_\omega(\chi) \cdot \bar{\nabla} u + \bar{\nabla} \text{tr}_\omega(\chi) \cdot \nabla u)}{A - u}\right) \\ & \quad - \epsilon \left(\frac{\nabla |\nabla u|_g^2 \cdot \bar{\nabla} u}{(A - u)^2} + \frac{\bar{\nabla} |\nabla u|_g^2 \cdot \nabla u}{(A - u)^2}\right) - 2\epsilon \frac{|\nabla u|_g^4}{(A - u)^3} - \frac{2(1 - \epsilon)}{A - u} \text{Re}(\nabla H \cdot \bar{\nabla} u) \\ & \quad + \frac{2(1 - \epsilon)}{A - u} \text{Re}(\nabla u \cdot \bar{\nabla}(\text{tr}_\omega(\chi))) + (\text{tr}_\omega(\chi) - \kappa) \frac{|\nabla u|_g^2}{(A - u)^2} + \left(\frac{\partial}{\partial t} - \Delta\right)\text{tr}_\omega(\chi) \end{aligned}$$



Since  $A - u$ ,  $(A - u)^{-1}$ ,  $tr_\omega(\chi)$  are uniformly bounded, by applying Lemma 2.2, Proposition 2.1, Proposition 2.2 and Schwarz inequality a few times, there exist  $C_1, C_2 > 0$  depending on  $\epsilon > 0$  such that

$$\begin{aligned}
 & \left( \frac{\partial}{\partial t} - \Delta \right) H \\
 \leq & \left( \frac{|\nabla u|_g^2 - |\nabla \nabla u|_g^2 - |\bar{\nabla} \nabla u|_g^2 + (\nabla tr_\omega(\chi) \cdot \bar{\nabla} u + \bar{\nabla} tr_\omega(\chi) \cdot \nabla u)}{A - u} \right) \\
 & - \epsilon \left( \frac{|\nabla \nabla u|_g^2 \cdot \bar{\nabla} u}{(A - u)^2} + \frac{|\bar{\nabla} \nabla u|_g^2 \cdot \nabla u}{(A - u)^2} \right) - 2\epsilon \frac{|\nabla u|_g^4}{(A - u)^3} - \frac{2(1 - \epsilon)}{A - u} Re(\nabla H \cdot \bar{\nabla} u) \\
 & + \frac{2(1 - \epsilon)}{A - u} Re(\nabla u \cdot \bar{\nabla}(tr_\omega(\chi))) + (tr_\omega(\chi) - \kappa) \frac{|\nabla u|_g^2}{(A - u)^2} - |\nabla tr_\omega(\chi)|^2 + C_1 \\
 \leq & -C_2 \epsilon |\nabla u|_g^4 - \frac{2 - 2\epsilon}{A - u} Re(\nabla H \cdot \bar{\nabla} u) + C_3.
 \end{aligned}$$

By applying the maximum principle at  $(z_0, t_0)$ , for  $H(z_0, t_0) = \max_{X \times [0, t]} H(t, z)$ ,  $H(z_0, t_0)$  is uniformly bounded and so is  $H(z, t)$  on  $X \times [0, \infty)$ . Then inequality (3.19) follows immediately.

We shall now prove inequality (3.20). Let

$$K = -\frac{\Delta u}{A - u} + \frac{4|\nabla u|_g^2}{A - u},$$

then the evolution for  $K$  is given by

$$\begin{aligned}
 & \left( \frac{\partial}{\partial t} - \Delta \right) K \\
 = & \left( \frac{4|\nabla u|_g^2 - \Delta u - 4|\nabla \nabla u|_g^2 - 3|\nabla \bar{\nabla} u|_g^2 + g^{i\bar{j}} g^{k\bar{l}} \chi_{i\bar{j}} \bar{\chi}_{k\bar{l}} - \Delta tr_\omega(\chi) + 8Re(\nabla tr_\omega(\chi) \cdot \bar{\nabla} u)}{A - u} \right) \\
 & + 4(tr_\omega(\chi) - \kappa) \frac{|\nabla u|_g^2}{(A - u)^2} - \frac{2}{A - u} Re(\nabla K \cdot \bar{u}) - (tr_\omega(\chi) - \kappa) \frac{\Delta u}{(A - u)^2}.
 \end{aligned}$$

On the other hand, we have

$$R_{i\bar{j}} = -u_{i\bar{j}} - \chi_{i\bar{j}}$$

and from Lemma 2.2 and Proposition 2.2, there exists  $C_4, C_5 > 0$  such that

$$\begin{aligned}
& -\Delta \operatorname{tr}_\omega(\chi) \\
&= \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr}_\omega(\chi) - \frac{\partial}{\partial t} \operatorname{tr}_\omega(\chi) \\
&= \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr}_\omega(\chi) - g^{i\bar{l}} g^{k\bar{j}} R_{k\bar{l}} \chi_{i\bar{j}} - \operatorname{tr}_\omega(\chi) \\
&= \left( \frac{\partial}{\partial t} - \Delta \right) \operatorname{tr}_\omega(\chi) + g^{i\bar{l}} g^{k\bar{j}} (u_{k\bar{l}} + \chi_{k\bar{l}}) \chi_{i\bar{j}} - \operatorname{tr}_\omega(\chi) \\
&\leq |\bar{\nabla} \nabla u|_g^2 - C_4 |\nabla \operatorname{tr}_\omega(\chi)|^2 + C_5.
\end{aligned}$$

Combining the above estimates with inequality (3.19) and applying Schwarz inequality, there exists  $C_6 > 0$  such that

$$\begin{aligned}
& \left( \frac{\partial}{\partial t} - \Delta \right) K \\
&\leq -\frac{|\nabla \bar{\nabla} u|_g^2}{A-u} - \frac{2}{A-u} \operatorname{Re}(\nabla K \cdot \bar{\nabla} u) + C_6 \\
&\leq -\frac{(\Delta u)^2}{A-u} - \frac{2}{A-u} \operatorname{Re}(\nabla K \cdot \bar{\nabla} u) + C_6.
\end{aligned}$$

By applying the maximum principle at  $(z_0, t_0)$  for  $K(z_0, t_0) = \max_{X \times [0, t]} K(z, t)$ ,  $K(z_0, t_0)$  is uniformly bounded and so is  $K(z, t)$  on  $X \times [0, \infty)$ . Then inequality (3.20) follows immediately.  $\square$

#### 4. PROOF OF THE MAIN THEOREMS

The scalar curvature  $R(t)$  along the normalized Kähler-Ricci flow (1.2) can be expressed by

$$(4.23) \quad R(t) = -\Delta u - \operatorname{tr}_\omega(\chi).$$

Now we can prove Theorem 1.1.

**Theorem 4.1.** *There exists  $C > 0$  such that on  $X \times [0, \infty)$ ,*

$$|R(t)| \leq C.$$

*Proof.* It is a well-known fact by the maximum principle that the scalar curvature  $R(t)$  is uniformly bounded from below. Hence it suffices to prove a uniform upper bound for  $R(t)$ , and it follows immediately from equation (4.23) and equation (3.20).  $\square$

Corollary 1.1 follows from Theorem 4.1 immediately by rescaling time and space.

The abundance conjecture holds for projective manifolds of dimension no bigger than three and by the classification of complex surfaces, the canonical line bundle of a Kähler surface is always semi-ample if it is nef. Therefore the canonical line bundle is semi-ample for Kähler surfaces and three dimensional projective manifolds of nef canonical line bundle. Corollary 1.2 then follows from Theorem 4.1.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854

*E-mail address:* `jiansong@math.rutgers.edu`

BICMR AND SMS, PEKING UNIVERSITY, BEIJING, 100871, CHINA

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544

*E-mail address:* `tian@math.princeton.edu`